

IONESCU'S THEOREM FOR HIGHER RANK GRAPHS

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ABSTRACT. We will define new constructions similar to the graph systems of correspondences described by Deaconu *et al.* We will use these to prove a version of Ionescu's theorem for higher rank graphs. Afterwards we will examine the properties of these constructions further and make contact with Yeend's topological k-graphs and the tensor-groupoid-valued product systems of Fowler and Sims.

1. INTRODUCTION

In [Ion07], Ionescu defines a natural correspondence associated to any *Mauldin-Williams* graph. A Mauldin-Williams graph is a directed graph with a compact metric space associated to each vertex and a contractive map associated to each edge (a more rigorous definition is presented below). These graphs generalize iterated function systems and have self-similar invariant sets. Ionescu proved that the Cuntz-Pimsner algebra of the correspondence associated to any Mauldin-Williams graph is isomorphic to the graph C^* -algebra of the underlying graph.

Here we prove an analogue for higher rank graphs. Our arguments make extensive use of the *graph systems of correspondences* construction presented in [DKPS10] and (we hope) provide an interesting application of their ideas. We also define some other systems similar to those [defined in] of [DKPS10] and briefly describe how all of these systems fit into what Fowler and Sims refer to in [FS02] as *product systems taking values in tensor groupoids*.

This paper is organized as follows. In Section 2 we will present a brief overview of some of the preliminaries on C^* -correspondences, graph systems of correspondences, and topological k-graph algebras. In Section 3 we will define two systems that closely resemble Λ -systems of correspondences which we will call Λ -systems of homomorphisms and Λ -systems of maps. The Λ -system of maps will be a generalization

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of the notion of a Mauldin-Williams graph. After defining some more terminology, we prove some basic facts about these systems and how they relate to one another. In Section 4 we define a k -graph analog of Mauldin-Williams graphs and prove our main theorem which generalizes Ionescu's main result from [Ion07]. In Section 5 we prove that the Cuntz-Pimsner algebra of the correspondence associated to any Λ -system of maps can be realized as the graph algebra of a certain topological k -graph. In Section 6 we briefly describe how all of the various Λ -systems fit in to the framework described by Fowler and Sims in [FS02]. In Section 7 we will examine the question of which Λ -systems of correspondences arise from the other types of Λ -systems described here. Finally, in Section 8 we show that, perhaps disappointingly, the higher-rank Mauldin-Williams graphs of Section 4 do not give rise to any new "higher-rank fractals".

2. PRELIMINARIES

2.1. Correspondences. For C^* -algebras A and B , we usually want our $A - B$ correspondences X to be *nondegenerate* in the sense that $A \cdot X = X$, equivalently, the left-module homomorphism $\varphi_A : A \rightarrow \mathcal{L}_B(X) = M(\mathcal{K}_B(X))$ is nondegenerate. If $\varphi : A \rightarrow M(B)$ is a nondegenerate homomorphism, the *standard $A - B$ correspondence* ${}_{\varphi}B$ is B viewed as a Hilbert module in the usual way and equipped with the left A -module structure induced by φ . The *identity B -correspondence* is ${}_{\text{id}}B$.

An *isomorphism* of an $A - B$ correspondence X onto a $C - D$ correspondence Y is a triple $(\theta_A, \theta, \theta_B)$, where

- $\theta_A : A \rightarrow C$ and $\theta_B : B \rightarrow D$ are C^* -isomorphisms, and
- $\theta : X \rightarrow Y$ is a linear bijection such that

$$\langle \theta(\xi), \theta(\eta) \rangle_D = \theta_B(\langle \xi, \eta \rangle_B) \quad \text{for all } \xi, \eta \in X;$$

$$\theta(a \cdot \xi \cdot b) = \theta_A(a) \cdot \theta(\xi) \cdot \theta_B(b) \quad \text{for all } a \in A, \xi \in X, b \in B.$$

θ_A and θ_B are the *left-* and *right-hand coefficient isomorphisms*, respectively. When both X and Y are $A - B$ correspondences, we require, unless otherwise specified, that the coefficient isomorphisms be the identity maps, and we sometimes emphasize that we are making this assumption by saying that $\theta : X \rightarrow Y$ is an *$A - B$ correspondence isomorphism*.

Recall from [EKQR00, Proposition 2.3] that for two nondegenerate homomorphisms $\varphi, \psi : A \rightarrow M(B)$, the standard $A - B$ correspondences ${}_{\varphi}B$ and ${}_{\psi}B$ are isomorphic if and only if there is a unitary multiplier $u \in M(B)$ such that $\psi = \text{Ad } u \circ \varphi$ (the special case of

imprimitivity bimodules is essentially [BGR77, Proposition 3.1]). In particular, if B is commutative then ${}_{\varphi}B \cong {}_{\psi}B$ if and only if $\varphi = \psi$.

2.2. Λ -systems. Throughout, Λ will be a row-finite k -graph with no sources, so that the associated Cuntz-Krieger relations take the most elementary form. In [DKPS10], Deaconu, Kumjian, Pask, and Sims introduced Λ -systems of correspondences: we have a Banach bundle $X \rightarrow \Lambda$ with fibres $\{X_{\lambda}\}_{\lambda \in \Lambda}$ such that

- (1) for each $v \in \Lambda^0$, X_v is a C^* -algebra;
- (2) for each $\lambda \in u\Lambda v$, X_{λ} is an $X_u - X_v$ correspondence;
- (3) there is a partially-defined associative multiplication on $X = \bigsqcup_{\lambda \in \Lambda} X_{\lambda}$ that is compatible with the multiplication in Λ via the bundle projection $X \rightarrow \Lambda$;
- (4) whenever $\lambda, \mu \notin \Lambda^0$ and $s(\lambda) = r(\mu)$, $x \otimes y \mapsto xy : X_{\lambda} \otimes_{A_{s(\lambda)}} X_{\mu} \rightarrow X_{\lambda\mu}$ is an isomorphism of $A_{r(\lambda)} - A_{s(\mu)}$ correspondences;
- (5) the left and right module multiplications of the correspondences coincide with the multiplication from the Λ -system.

For a Λ -system X of correspondences, we will write

$$\varphi_{\lambda} : X_u \rightarrow \mathcal{L}(X_{\lambda}) \quad \text{for } \lambda \in u\Lambda$$

for the left-module structure map. Note that the multiplication in X induces $X_u - X_v$ correspondence isomorphisms $X_{\lambda} \otimes X_v \cong X_{\lambda}$ for all $\lambda \in u\Lambda v$, but only induces isomorphisms $X_u \otimes X_{\lambda} \cong X_{\lambda}$ if every correspondence X_{λ} is nondegenerate.

Given Λ -systems X and Y of correspondences, a map $\theta : X \rightarrow Y$ is a Λ -system isomorphism if

- (1) for all $\lambda \in u\Lambda v$,

$$\theta_{\lambda} := \theta|_{X_{\lambda}} : X_{\lambda} \rightarrow Y_{\lambda}$$

is an isomorphism of correspondences with coefficient isomorphisms θ_u, θ_v ;

- (2) for all $\lambda \in u\Lambda v, \mu \in v\Lambda w$,

$$\theta_{\lambda}(\xi)\theta_{\mu}(\eta) = \theta_{\lambda\mu}(\xi\eta) \quad \text{for all } \xi \in X_{\lambda}, \eta \in X_{\mu}.$$

Since the multiplication in the Λ -system induces the left and right module multiplications for the correspondences, in the above we can relax (1) to

- (1)' for all $\lambda \in \Lambda v$, $\theta_{\lambda} : X_{\lambda} \rightarrow Y_{\lambda}$ is a linear bijection satisfying

$$\langle \theta_{\lambda}(\xi), \theta_{\lambda}(\eta) \rangle_{Y_v} = \theta_v(\langle \xi, \eta \rangle_{X_v}) \quad \text{for all } \xi, \eta \in X_{\lambda},$$

because (2) takes care of the coefficient maps. We emphasize that we're requiring that, for each $v \in \Lambda^0$, θ_v be the right-hand coefficient

isomorphism for every correspondence isomorphism θ_λ with $s(\lambda) = v$, and also the left-hand coefficient isomorphism for every correspondence isomorphism θ_λ with $r(\lambda) = v$. Thus, if X and Y are isomorphic Λ -systems of correspondences, then without loss of generality we may assume (if we wish) that $X_v = Y_v$ and $\theta_v = \text{id}_{X_v}$ for every vertex v , so that $\theta_\lambda : X_\lambda \rightarrow Y_\lambda$ is an $X_u - X_v$ correspondence isomorphism whenever $\lambda \in u\Lambda v$.

2.3. Topological k -graphs. Recall [Yee06] that a *topological k -graph* is a k -graph Γ equipped with a locally compact Hausdorff topology making the multiplication continuous and open, the range map continuous, the source map a local homeomorphism, and the degree functor $d : \Gamma \rightarrow \mathbb{N}^k$ continuous. Carlsen, Larsen, Sims, and Vittadello show in [CLSV11, Proposition 5.9] that every topological k -graph Γ gives rise to a \mathbb{N}^k -system Z of correspondences over $A := C_0(\Gamma^0)$ as follows: For each $n \in \mathbb{N}^k$ let Z_n be the A -correspondence associated to the topological graph $(\Gamma^0, \Gamma^n, s|_{\Gamma^n}, r|_{\Gamma^n})$, so that Z_n is the completion of the pre-correspondence $C_c(\Gamma^n)$ with operations

$$(f \cdot \xi \cdot g)(\alpha) = f(r(\alpha))\xi(\alpha)g(s(\alpha))$$

$$\langle \xi, \eta \rangle_A(v) = \sum_{\alpha \in \Gamma^n v} \overline{\xi(\alpha)}\eta(\alpha),$$

for $\xi, \eta \in C_c(\Gamma^n)$, $f, g \in A$. Then for $\xi \in C_c(\Gamma^n)$, $\eta \in C_c(\Gamma^m)$ define $\xi\eta \in C_c(\Gamma^{n+m})$ by

$$(\xi\eta)(\alpha\beta) = \xi(\alpha)\eta(\beta) \quad \text{for } \alpha \in \Gamma^n, \beta \in \Gamma^m, s(\alpha) = r(\beta).$$

In [Yee06], Yeeend defined $C^*(\Gamma)$ using a groupoid model, but [CLSV11, Theorem 5.20] shows that $C^*(\Gamma) \cong \mathcal{NO}_Z$, where \mathcal{NO}_Z is the Cuntz-Nica-Pimsner algebra of the product system Z . The topological k -graphs we encounter in this paper will be nice enough that \mathcal{NO}_Z will coincide with the Cuntz-Pimsner algebra \mathcal{O}_Z .

3. OTHER Λ -SYSTEMS

We introduce a few constructions that are similar to Λ -systems of correspondences:

Definition 3.1. (1) A *Λ -system of homomorphisms* is a pair (\mathcal{A}, φ) , where $\mathcal{A} \rightarrow \Lambda^0$ is a C^* -bundle and for each $\lambda \in u\Lambda v$ we have a nondegenerate homomorphism $\varphi_\lambda : A_u \rightarrow M(A_v)$, such that

$$\varphi_\mu \circ \varphi_\lambda = \varphi_{\lambda\mu} \quad \text{if } s(\lambda) = r(\mu)$$

$$\varphi_v = \text{id}_{A_v} \quad \text{for } v \in \Lambda^0,$$

where we have canonically extended φ_μ to $M(A_v)$.

- (2) A Λ -system of maps is a pair (T, σ) , where $T \rightarrow \Lambda^0$ is a bundle of locally compact Hausdorff spaces and for each $\lambda \in u\Lambda v$ we have a continuous map $\sigma_\lambda : T_v \rightarrow T_u$, such that

$$\begin{aligned} \sigma_\lambda \circ \sigma_\mu &= \sigma_{\lambda\mu} \quad \text{if } s(\lambda) = r(\mu) \\ \sigma_v &= \text{id}_{T_v} \quad \text{for } v \in \Lambda^0. \end{aligned}$$

Remark 3.2. (1) Note that we need to impose the nondegeneracy condition on the homomorphisms φ_λ so that composition is defined.

- (2) Thus, a Λ -system of homomorphisms is essentially a contravariant functor from Λ to the category of C^* -algebras and nondegenerate homomorphisms into multiplier algebras, and a Λ -system of maps is essentially a (covariant) functor from Λ to the category of locally compact Hausdorff spaces and continuous maps.
- (3) Every Λ -system (T, σ) of maps gives rise to a Λ -system (\mathcal{A}, σ^*) of homomorphisms, with

$$\begin{aligned} A_v &= C_0(T_v) \quad \text{for } v \in \Lambda^0 \\ \sigma_\lambda^*(f) &= f \circ \sigma_\lambda \quad \text{for } \lambda \in \Lambda, f \in A_{r(\lambda)}. \end{aligned}$$

- (4) Every Λ -system (\mathcal{A}, φ) of homomorphisms gives rise to a Λ -system of correspondences: for $\lambda \in u\Lambda v$ let X_λ be the standard $A_u - A_v$ correspondence $\varphi_\lambda A_v$.

Definition 3.3. We call a Λ -system of maps (T, σ)

- (1) *proper* if each map $\sigma_\lambda : T_{s(\lambda)} \rightarrow T_{r(\lambda)}$ is proper (in the usual sense that inverse images of compact sets are compact), and
- (2) *dense* if each map $\sigma_\lambda : T_{s(\lambda)} \rightarrow T_{r(\lambda)}$ has dense range.

Definition 3.4. We call a C^* -homomorphism $\varphi : A \rightarrow M(B)$ *proper* if it maps into B (and we will also denote it by $\varphi : A \rightarrow B$).

Remark 3.5. A nondegenerate homomorphism $\varphi : A \rightarrow M(B)$ is proper in the above sense if and only if φ takes one (hence every) bounded approximate identity for A to an approximate identity for B . Also, if $\sigma : Y \rightarrow X$ is a continuous map, then $\sigma^* : C_0(X) \rightarrow M(C_0(Y))$ is automatically nondegenerate, and is proper if and only if σ is proper.

Definition 3.6. Let X an $A - B$ correspondence, with left module map $\varphi_A : A \rightarrow \mathcal{L}(X) = M(\mathcal{K}(X))$. We call X *proper*, *nondegenerate*, or *faithful* if φ_A has the corresponding property.

Definition 3.7. We call a Λ -system (\mathcal{A}, φ) of homomorphisms *proper* or *faithful* if each homomorphism φ_λ has the corresponding property.

Definition 3.8. We call a Λ -system X of correspondences *proper*, *non-degenerate*, *full*, or *faithful* if each correspondence X_λ has the corresponding property.

[DKPS10] calls a Λ -system X of correspondences *regular* if it is proper, nondegenerate, full, and faithful. However, we believe that the fidelity is too much to ask, both for aesthetic and practical reasons.

Let X be a Λ -system of correspondences, and let $A = \bigoplus_{v \in \Lambda^0} X_v$ be the c_0 -direct sum of C^* -algebras. Then each X_λ may be regarded as an A -correspondence. For each $n \in \mathbb{N}^k$, [DKPS10] defines an A -correspondence Y_n by

$$Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda,$$

and [DKPS10, Proposition 3.17] shows that $Y = Y_X = \bigsqcup_{n \in \mathbb{N}^k} Y_n$ is an \mathbb{N}^k -system (i.e., a product system over \mathbb{N}^k) of A -correspondences, and moreover if X is regular then so is Y . We will identify X_λ with its canonical image in $Y_{d(\lambda)}$, i.e., we will blur the distinction between the external and internal direct sums of the A -correspondences $\{X_\lambda : \lambda \in \Lambda^n\}$.

Definition 3.9. We call a Λ -system (T, σ) of maps

- (1) *k-dense* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$T_v = \overline{\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})},$$

and

- (2) *k-regular* if it is proper and *k-dense*.

Here is a minor strengthening of *k-density* that we will find useful later:

Definition 3.10. A Λ -system of maps (T, σ) is *k-surjective* if

$$T_v = \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)}) \quad \text{for all } v \in \Lambda^0, n \in \mathbb{N}^k.$$

Definition 3.11. We call a Λ -system (\mathcal{A}, φ) of homomorphisms

- (1) *k-faithful* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$\bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda = \{0\},$$

and

- (2) *k-regular* if it is proper and *k-faithful*.

Definition 3.12. We call a Λ -system X of correspondences

- (1) *k-faithful* if the associated \mathbb{N}^k -system Y_X is faithful, and
- (2) *k-regular* if it is proper, nondegenerate, full, and *k-faithful*.

Remark 3.13. (1) If (T, σ) is a Λ -system of maps, then the associated Λ -system (\mathcal{A}, σ^*) of homomorphisms is:

- proper if and only if (T, σ) is, and
- faithful if and only if (T, σ) is dense.

- (2) If (\mathcal{A}, φ) is a Λ -system of homomorphisms, then the associated Λ -system X of correspondences is:

- automatically nondegenerate and full, and
- proper or faithful if and only if (\mathcal{A}, φ) has the corresponding property.

- (3) We have organized our definitions so that a Λ -system X of correspondences is *k-regular* if and only if the associated \mathbb{N}^k -system Y_X is regular.

We will need the following variation on the above:

Lemma 3.14. (1) *If (T, σ) is a Λ -system of maps, then the associated Λ -system (\mathcal{A}, σ^*) of homomorphisms is *k-faithful* if and only if (T, σ) is *k-dense*, and consequently is *k-regular* if and only if (T, σ) is.*

- (2) *If (\mathcal{A}, φ) is a Λ -system of homomorphisms, then the associated Λ -system X of correspondences is *k-faithful* if and only if (\mathcal{A}, φ) is, and consequently is *k-regular* if and only if (X, φ) is.*

Proof. (1). This is routine, but we present the details for completeness. First assume that (T, σ) is not *k-dense*. We will show that (\mathcal{A}, σ^*) is not *k-faithful*. We can choose $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$ is not dense in T_v . We will show that $\bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^* \neq \{0\}$. We can choose a nonzero $f \in C_0(T_v)$ that vanishes on $\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$. Then for all $\lambda \in v\Lambda^n$ and all $g \in C_0(T_{s(\lambda)})$,

$$\sigma_\lambda^*(f)g = (f \circ \sigma_\lambda)g = 0.$$

Thus $f \in \bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^*$.

Conversely, assume that (\mathcal{A}, σ^*) is not *k-faithful*. We will show that (T, σ) is not *k-dense*. We can choose $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $\bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^* \neq \{0\}$. We will show that $\bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$ is not dense in T_v . Choose a nonzero $f \in \bigcap_{\lambda \in v\Lambda^n} \ker \sigma_\lambda^*$. Then choose a nonempty open set $U \subset T_v$ such that $f(t) \neq 0$ for all $t \in U$. We will show that

$$U \cap \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)}) = \emptyset.$$

Let $t \in \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)})$, and choose $\lambda \in v\Lambda^n$ and $s \in T_{s(\lambda)}$ such that $t = \sigma_\lambda(s)$. Then choose $g \in C_0(T_{s(\lambda)})$ such that $g(s) = 1$. Since $f \in \ker \sigma_\lambda^*$,

$$0 = (\sigma_\lambda^*(f)g)(s) = f(\sigma_\lambda(s))g(s) = f(t),$$

so $t \notin U$.

(2). First assume that (\mathcal{A}, φ) is not k -faithful. We will show that X is not k -faithful. We can choose $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $\bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda \neq \{0\}$. We will show that the A -correspondence Y_n is not faithful. Choose a nonzero $a \in A_v$ such that $\varphi_\lambda(a) = 0$ for all $\lambda \in v\Lambda^n$. Let

$$y = (x_\lambda) \in Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda.$$

Then $a \cdot y$ is the Λ^n -tuple $(a \cdot x_\lambda)$, where for $\lambda \in \Lambda^n$ we have

$$a \cdot x_\lambda = \begin{cases} \varphi_\lambda(a)x_\lambda & \text{if } r(\lambda) = v \\ 0 & \text{if } r(\lambda) \neq v. \end{cases}$$

Since $\varphi_\lambda(a)x_\lambda = 0$ for all $\lambda \in v\Lambda^n$, $x_\lambda \in X_\lambda = A_{s(\lambda)}$, we have $a \cdot y = 0$, and we have shown that Y_n is not faithful.

Conversely, assume that X is not k -faithful. We will show that (\mathcal{A}, φ) is not k -faithful. We can choose $n \in \mathbb{N}^k$ such that the A -correspondence Y_n is not faithful, so we can find a nonzero $a \in A$ such that $a \cdot y = 0$ for all $y \in Y_n$. Then $a = (a_v)$ is a Λ^0 -tuple with $a_v \in A_v$ for each v , so we can choose $v \in \Lambda^0$ such that $a_v \neq 0$. We will show that $a_v \in \bigcap_{\lambda \in v\Lambda^n} \ker \varphi_\lambda$. Let $\lambda \in v\Lambda^n$ and $b \in A_{s(\lambda)}$. Define a $v\Lambda^n$ -tuple $(x_\mu) \in Y_n$ by

$$x_\mu = \begin{cases} b & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda. \end{cases}$$

Then

$$\varphi_\lambda(a_v)b = (a_v \cdot (x_\lambda))_\lambda = 0. \quad \square$$

Remark 3.15. The argument in the last paragraph of the above proof is a routine adaptation of that used in [DKPS10, Proposition 3.1.7].

Our motivation for introducing the properties of k -density and k -fidelity is that the Mauldin-Williams graphs considered by Ionescu — where we have a 1-graph Λ whose 1-skeleton E is finite, a Λ -system (T, σ) of maps in which each space T_v is a compact metric space and each map σ_λ is a (strict) contraction — are typically 1-dense in the above sense rather than dense. More precisely, a Mauldin-Williams graph (T, σ) is dense (in our terminology) if and only if every map σ_e

(for $e \in E^1$) is surjective, which is usually not the case, and 1-dense if and only if for all $v \in E^0$ we have

$$\bigcup_{e \in vE^1} \sigma_e(T_{s(e)}) = T(v),$$

which is always the case (after replacing the original spaces by an “invariant list”). Thus, since we want to consider a version of Ionescu’s theorem for k -graphs, we must allow the weakened property of k -fidelity (of Definition 3.12) rather than insisting upon fidelity.

[DKPS10, Definition 3.2.1] defines a *representation* of a Λ -system X in a C^* -algebra B as a map $\rho : X \rightarrow B$ such that

- (1) for each $v \in \Lambda^0$, $\rho_v : X_v \rightarrow B$ is a C^* -homomorphism;
- (2) whenever $\xi \in X_\lambda, \eta \in X_\mu$,

$$\rho_\lambda(\xi)\rho_\mu(\eta) = \begin{cases} \rho_{\lambda\mu}(\xi\eta) & \text{if } s(\lambda) = r(\mu) \\ 0 & \text{if otherwise;} \end{cases}$$

- (3) whenever $\xi \in X_\lambda, \eta \in X_\mu$, if $d(\lambda) = d(\mu)$ then

$$\rho_\lambda(\xi)^*\rho_\mu(\eta) = \begin{cases} \rho_{s(\lambda)}(\langle \xi, \eta \rangle_{X_{s(\lambda)}}) & \text{if } \lambda = \mu \\ 0 & \text{if otherwise,} \end{cases}$$

and when X is regular [DKPS10] defines a representation ρ to be *Cuntz-Pimsner covariant* if for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$, and $a \in X_v$,

- (4)

$$\rho_v(a) = \sum_{\lambda \in v\Lambda^n} \rho^{(\lambda)}(\varphi_\lambda(a)),$$

where $\rho^{(\lambda)} = \rho_\lambda^{(1)} : \mathcal{K}(X_\lambda) \rightarrow B$ is the associated homomorphism. Then [DKPS10] defines a representation ρ to be *universal* if for every representation $\tau : X \rightarrow C$ there is a unique C^* -homomorphism $\Phi = \Phi_\tau : B \rightarrow C$ such that $\Phi \circ \rho_\lambda = \tau_\lambda$ for all $\lambda \in \Lambda$, and a Cuntz-Pimsner covariant representation to be *universal* if it satisfies the above universality property for all Cuntz-Pimsner covariant representations. Then [DKPS10] points out that, by the nondegeneracy that is included in the regularity assumption, (1)–(3) above can be replaced by the following set of conditions: each ρ_λ is a correspondence representation in B , ρ is multiplicative whenever this makes sense, and ρ_u and ρ_v have orthogonal images for all $u \neq v \in \Lambda^0$.

For the \mathbb{N}^k -system $Y = Y_X$ associated to a regular Λ -system X , [DKPS10, Proposition 3.2.3] shows that there is a bijection between the representations $\rho : X \rightarrow B$ and the representations $\psi : Y \rightarrow B$

such that

$$\psi \circ \iota_\lambda = \rho_\lambda \quad \text{for all } \lambda \in \Lambda.$$

However, it is crucial for our results to note that the proof of [DKPS10, Proposition 3.2.3] only requires nondegeneracy of Y , not of X .

[DKPS10, Proposition 3.2.5] shows that if X is regular then a representation $\rho : X \rightarrow B$ is Cuntz-Pimsner covariant if and only if the associated representation $\psi : Y \rightarrow B$ is. We turn this result into a *definition*:

Definition 3.16. Let X be a k -regular Λ -system of correspondences, with associated \mathbb{N}^k -system Y , and let $\rho : X \rightarrow B$ be a representation of X , with associated representation $\psi : Y \rightarrow B$. We define ρ to be *Cuntz-Pimsner covariant* if ψ is, in other words

$$\sum_{\lambda \in v\Lambda^n} \rho^{(\lambda)} \circ \varphi_\lambda = \rho_v \quad \text{for all } v \in \Lambda^0.$$

Remark 3.17. To reiterate, the only difference between Definition 3.16 and the definition of Cuntz-Pimsner covariance given in [DKPS10, Definition 3.2.1] is that in the latter the Λ -system X is required to be regular, while we only require k -regularity. In any event, [DKPS10, Definition 3.2.7] defines the C^* -algebra of a regular Λ -system X to be the Cuntz-Pimsner algebra \mathcal{O}_Y , and in [DKPS10, Corollary 3.2.6] they notice that the representation $\rho^{j_Y} : X \rightarrow \mathcal{O}_Y$ is a universal Cuntz-Pimsner covariant representation, where $j_Y : Y \rightarrow \mathcal{O}_Y$ is the universal Cuntz-Pimsner covariant representation.

We emphasize that, even though we only require the Λ -system X to be k -regular, the theory of [DKPS10] carries over with no problems, as we've indicated. [DKPS10] uses the notation $C^*(A, X, \chi)$ for the C^* -algebra of X , but we'll write it as \mathcal{O}_X . If $\rho : X \rightarrow B$ is any Cuntz-Pimsner covariant representation, we'll write $\Phi_\rho : \mathcal{O}_X \rightarrow B$ for the homomorphism whose existence is guaranteed by universality; [DKPS10] would write it as $\Phi_{\rho, \pi}$, because they write π for the restriction of ρ to the C^* -bundle $X|_{\Lambda^0}$ (and they write A for this C^* -bundle, as well as for the section algebra $\bigoplus_{v \in \Lambda^0} X_v$ — we reserve the name A for this latter C^* -algebra).

Note that since we assume that Λ is row-finite with no sources, the infinite-path space Λ^∞ is locally compact Hausdorff, and is the disjoint union of the compact open subsets $\{v\Lambda^\infty\}_{v \in \Lambda^0}$. We get a Λ -system of maps (Λ^∞, τ) , where for $\lambda \in u\Lambda v$ the continuous map

$$\tau_\lambda : v\Lambda^\infty \rightarrow u\Lambda^\infty$$

is defined by $\tau_\lambda(x) = \lambda x$. Moreover, this Λ -system is k -regular. This system has the following properties: if $\lambda \in u\Lambda v$ then τ_λ is a homeomorphism of $v\Lambda^\infty$ onto the compact open set

$$\lambda\Lambda^\infty \subset u\Lambda^\infty,$$

and consequently τ_λ^* is a surjection of $C(u\Lambda^\infty)$ onto $C(v\Lambda^\infty)$.

Lemma 3.18. *For each $u \in \Lambda^0$ and $\lambda \in u\Lambda$ let $p_\lambda = s_\lambda s_\lambda^*$, the set*

$$D_u = \overline{\text{span}}\{p_\lambda : \lambda \in u\Lambda\}$$

is a unital commutative C^ -subalgebra of $C^*(\Lambda)$, with unit p_u , and the subalgebras $\{D_u\}_{u \in \Lambda^0}$ are pairwise orthogonal. Moreover, if D denotes the commutative C^* -subalgebra $\bigoplus_{u \in \Lambda^0} D_u$ of $C^*(\Lambda)$, then there is an isomorphism $\theta : C_0(\Lambda^\infty) \rightarrow D$ that takes the characteristic function of $\lambda\Lambda^\infty = \{\lambda x : s(\lambda) = r(x)\}$ to p_λ and $C(u\Lambda^\infty)$ to D_u . Finally, the diagram*

$$\begin{array}{ccc} C(u\Lambda^\infty) & \xrightarrow{\tau_\lambda^*} & C(v\Lambda^\infty) \\ \theta \downarrow & & \downarrow \theta \\ D_u & \xrightarrow{\text{Ad } s_\lambda^*} & D_v \end{array}$$

commutes.

Proof. This is probably folklore, at least for directed graphs, and in any case is standard: truncation gives an inverse system $\{\Lambda^n, \tau_{m,n}\}$ of surjections among nonempty finite sets¹, whose inverse limit is Λ^∞ , and for each n the commutative C^* -subalgebra $D^n := \overline{\text{span}}\{p_\lambda : \lambda \in \Lambda^n\}$ of $C^*(\Lambda)$ has spectrum Λ^n , so the inductive limit $D = \overline{\text{span}}\{D^n : n \in \mathbb{N}^k\}$ has spectrum Λ^∞ . \square

Definition 3.19. Let (T, σ) be a Λ -system of maps. A continuous map $\Phi : \Lambda^\infty \rightarrow T$ is *intertwining* if

$$\Phi \circ \tau_\lambda = \sigma_\lambda \circ \Phi \quad \text{for all } \lambda \in \Lambda.$$

We say (T, σ) is *self-similar* if there is a surjective intertwining map $\Phi : \Lambda^\infty \rightarrow T$.

Proposition 3.20. *Every self-similar Λ -system of maps (T, σ) is k -surjective, and each space T_v is compact.*

Proof. First, T_v is compact because the intertwining property and surjectivity of Φ imply that $T_v = \Phi(v\Lambda^\infty)$, which is a continuous image

¹with $\tau_{m,n}(\lambda) = \lambda(0, n)$ for $\lambda \in \Lambda^m$ and $n \leq m$

of the compact set $v\Lambda^\infty$. For the k -surjectivity, if $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ then

$$\begin{aligned}
T_v &= \Phi(v\Lambda^\infty) \\
&= \Phi\left(\bigcup_{\lambda \in v\Lambda^n} \lambda\Lambda^\infty\right) \\
&= \bigcup_{\lambda \in v\Lambda^n} \Phi(\lambda\Lambda^\infty) \\
&= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(\Phi(s(\lambda)\Lambda^\infty)) \\
&= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(T_{s(\lambda)}). \quad \square
\end{aligned}$$

Definition 3.21. Let (T, σ) be a Λ -system of maps, and let $S \subset T$ be locally compact. For each $v \in \Lambda^0$ let $S_v = S \cap T_v$. Suppose that

$$\sigma_\lambda(S_v) \subset S_u \quad \text{whenever } \lambda \in u\Lambda v.$$

Then $(S, \sigma|_S)$ is a Λ -subsystem of (T, σ) , where

$$(\sigma|_S)_\lambda = \sigma_\lambda|_{S_v} \quad \text{for all } \lambda \in \Lambda_v.$$

Note that our terminology makes sense: every Λ -subsystem is in fact a Λ -system.

Proposition 3.22. Let (T, σ) be a Λ -system of maps, and let $\Phi : \Lambda^\infty \rightarrow T$ be an intertwining map. Put

$$\begin{aligned}
T'_v &= \Phi(v\Lambda^\infty) \quad \text{for each } v \in \Lambda^0 \\
T' &= \bigcup_{v \in \Lambda^0} T'_v.
\end{aligned}$$

Then $(T', \sigma|_{T'})$ is a self-similar k -surjective Λ -subsystem of (T, σ) , and each T'_v is compact.

Proof. First of all, each T'_v is compact since $v\Lambda^\infty$ is compact and T_v is Hausdorff. Thus T' is locally compact, since the sets T'_v are pairwise disjoint and open. For each $\lambda \in u\Lambda v$ we have

$$\begin{aligned}
\sigma_\lambda(T'_v) &= \sigma_\lambda(\Phi(v\Lambda^\infty)) \\
&= \Phi(\lambda\Lambda^\infty) \\
&\subset \Phi(u\Lambda^\infty) \\
&= T'_u,
\end{aligned}$$

so $(T', \sigma|_{T'})$ is a Λ -subsystem of (T, σ) . It is self-similar because Φ is intertwining and maps Λ^∞ onto T' by construction. Then by Proposition 3.20 $(T', \sigma|_{T'})$ is k -surjective. \square

Theorem 3.23. *Let (T, σ) be a self-similar k -regular Λ -system of maps, and let X be the associated Λ -system of correspondences. Then*

$$\mathcal{O}_X \cong C^*(\Lambda).$$

Proof. Our strategy will be to find a Cuntz-Pimsner covariant representation $\rho : X \rightarrow C^*(\Lambda)$ whose image contains the generators, and then apply the Gauge-Invariant Uniqueness Theorem. Recall that for $\lambda \in u\Lambda v$, X_λ is the standard $C_0(T_u) - C_0(T_v)$ correspondence $\sigma_\lambda^* C_0(T_v)$. Define $\rho_\lambda : X_\lambda \rightarrow C^*(\Lambda)$ by

$$\rho_\lambda(f) = s_\lambda \theta \circ \Phi^*(f) \quad \text{for } f \in C_0(T_v).$$

Then ρ_λ is linear, and for $f, g \in C_0(T_v)$ we have

$$\begin{aligned} \rho_\lambda(f)^* \rho_\lambda(g) &= \theta(\Phi^*(\bar{f})) s_\lambda^* s_\lambda \theta(\Phi^*(g)) \\ &= \theta(\Phi^*(\bar{f})) p_v \theta(\Phi^*(g)) \\ &= \theta(\Phi^*(\bar{f}g)) \\ &= p_v(\langle f, g \rangle_{C(T_v)}). \end{aligned}$$

For $\lambda \in \Lambda v$, $\mu \in v\Lambda w$, $f \in C_0(T_v)$, and $h \in C_0(T_w)$ we have

$$\begin{aligned} \rho_\lambda(f) \rho_\mu(h) &= s_\lambda \theta(\Phi^*(f)) s_\mu \theta(\Phi^*(h)) \\ &= s_\lambda \theta(\Phi^*(f)) p_\mu s_\mu \theta(\Phi^*(h)) \\ &= s_\lambda p_\mu \theta(\Phi^*(f)) s_\mu \theta(\Phi^*(h)) \\ &= s_\lambda s_\mu \text{Ad } s_\mu^* \circ \theta(\Phi^*(f)) \theta(\Phi^*(h)) \\ &= s_{\lambda\mu} \theta \circ \tau_\mu^*(\Phi^*(f)) \theta(\Phi^*(h)) \\ &= s_{\lambda\mu} \theta(\tau_\mu^* \circ \Phi^*(f)) \theta(\Phi^*(h)) \\ &= s_{\lambda\mu} \theta(\Phi^* \circ \sigma_\mu^*(f)) \theta(\Phi^*(h)) \\ &= s_{\lambda\mu} \theta(\Phi^*(\sigma_\mu^*(f)h)) \\ &= \rho_{\lambda\mu}(\sigma_\mu^*(f)h) \\ &= \rho_{\lambda\mu}(fh). \end{aligned}$$

It follows that $\rho : X \rightarrow C^*(\Lambda)$ is a representation.

Next we show that ρ is Cuntz-Pimsner covariant. Let $u \in \Lambda^0$, $n \in \mathbb{N}^k$, and $f \in X_u = C_0(T_u)$. We need to show that

$$\rho_u(f) = \sum_{\lambda \in u\Lambda^n} \rho^{(\lambda)} \circ \varphi_\lambda(f),$$

where

$$\varphi_\lambda : C_0(T_u) \rightarrow \mathcal{K}(X_\Lambda)$$

is the left-module structure map. We need a little more information regarding the homomorphism

$$\rho^{(\lambda)} = \rho_\lambda^{(1)} : \mathcal{K}(X_\lambda) \rightarrow C^*(\Lambda).$$

For $\lambda \in u\Lambda v$ we have $X_\lambda = \sigma_\lambda^* C_0(T_v)$, so

$$\mathcal{K}(X_\lambda) = C_0(T_v),$$

and for $g, h \in C_0(T_v)$ the rank-one operator $\theta_{g,h}$ is given by (left) multiplication by $g\bar{h}$. Thus

$$\begin{aligned} \rho^{(\lambda)}(g\bar{h}) &= \rho_\lambda(g)\rho_\lambda(h)^* \\ &= s_\lambda \theta \circ \Phi^*(g) \theta \circ \Phi^*(\bar{h}) s_\lambda^* \\ &= s_\lambda \circ \theta \circ \Phi^*(g\bar{h}) s_\lambda^* \\ &= \text{Ad } s_\lambda \circ \rho_v(g\bar{h}). \end{aligned}$$

Since every function in $C_0(T_v)$ can be factored as $g\bar{h}$, we conclude that the homomorphism $\rho^{(\lambda)}$ coincides with

$$\text{Ad } s_\lambda \circ \rho_v : C_0(T_v) \rightarrow C^*(\Lambda).$$

Also, $\varphi_\lambda : C_0(T_u) \rightarrow \mathcal{K}(X_\Lambda)$ coincides with $\sigma_\lambda^* : C_0(T_u) \rightarrow C_0(T_v)$ (note that σ_λ^* maps into $C_0(T_v)$ because σ_λ is proper). Thus

$$\begin{aligned} \sum_{\lambda \in u\Lambda^n} \rho^{(\lambda)} \circ \varphi_\lambda(f) &= \sum_{\lambda \in u\Lambda^n} \text{Ad } s_\lambda \circ \theta \circ \Phi^* \circ \sigma_\lambda^*(f) \\ &= \sum_{\lambda \in u\Lambda^n} \text{Ad } s_\lambda \circ \theta \circ \tau_\lambda^* \circ \Phi^*(f) \\ &= \sum_{\lambda \in u\Lambda^n} \text{Ad } s_\lambda \circ \text{Ad } s_\lambda^* \circ \theta \circ \Phi^*(f) \\ &= \sum_{\lambda \in u\Lambda^n} \text{Ad } s_\lambda s_\lambda^* \circ \theta \circ \Phi^*(f) \\ &= \sum_{\lambda \in u\Lambda^n} p_\lambda \theta \circ \Phi^*(f) \\ &= p_u \rho_u(f) \quad \left(\text{since } \sum_{\lambda \in u\Lambda^n} p_\lambda = p_u \right) \\ &= \rho_u(f), \end{aligned}$$

since $\rho_u(C_0(T_u)) \subset D_u$ and $p_u = 1_{D_u}$.

Therefore ρ gives rise to a homomorphism $\Psi_\rho : \mathcal{O}_X \rightarrow C^*(\Lambda)$ such that

$$\Psi_\rho \circ \rho^X = \rho,$$

where $\rho^X : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is the universal Cuntz-Pimsner covariant representation. For each $v \in \Lambda^0$, the continuous map $\Phi : \Lambda^\infty \rightarrow T$ takes $v\Lambda^\infty$ into T_v , so Φ^* restricts to a nondegenerate homomorphism from $C_0(T_v)$ to $C(v\Lambda^\infty)$, and hence the homomorphism $\rho_v : C_0(T_v) \rightarrow D_v$ is nondegenerate. It follows that for each $\lambda \in \Lambda v$ the generator s_λ is in the range of $\rho_\lambda : X_\lambda \rightarrow C^*(\Lambda)$. Thus $\Psi_\rho : \mathcal{O}_X \rightarrow C^*(\Lambda)$ is surjective.

Finally, we appeal to the Gauge-Invariant Uniqueness Theorem [DKPS10, Theorem 3.3.1] to show that Ψ_ρ is injective. Note that [DKPS10] assume that the Λ -system X is regular, while we only assume that it is k -regular; as we have mentioned before, k -regularity is all that's required to make the results of [DKPS10] true. First of all, for each $v \in \Lambda^0$, Φ maps $v\Lambda^0$ onto T_v , and it follows that $\rho_v : C_0(T_v) \rightarrow D_v$ is faithful. Thus the direct sum

$$\Psi_\rho|_A = \bigoplus_{v \in \Lambda^0} \rho_v : \bigoplus_{v \in \Lambda^0} C_0(T_v) \rightarrow \bigoplus_{v \in \Lambda^0} D_v \subset C^*(\Lambda)$$

is also faithful. Let $\gamma : \mathbb{T}^k \rightarrow \text{Aut } C^*(\Lambda)$ be the gauge action. For $\lambda \in \Lambda^n v$, $f \in C_0(T_v)$, and $z \in \mathbb{T}^k$,

$$\begin{aligned} \gamma_z \circ \rho_\lambda(f) &= \gamma_z(s_\lambda \theta \circ \Phi^*(f)) \\ &= \gamma_z(s_\lambda) \rho_v(f) \quad (\text{since } \rho_v(f) \in D_v \subset C^*(\Lambda)^\gamma) \\ &= z^n s_\lambda \rho_v(f) \\ &= z^n \rho_\lambda(f), \end{aligned}$$

so by [DKPS10, Theorem 3.3.1] Ψ_ρ is faithful. \square

4. MAULDIN-WILLIAMS k -GRAPHS

We continue to let Λ be a row-finite k -graph with no sources.

Proposition 4.1. *Let (T, σ) be a Λ -system of maps such that each T_v is a complete metric space and, for some $c < 1$ and every $\lambda \in \Lambda$,*

$$\delta_v(\sigma_\lambda(t), \sigma_\lambda(s)) \leq c^{|\mathbf{d}(\lambda)|} \delta_v(t, s) \quad \text{for all } \lambda \in \Lambda, t, s \in T_{s(\lambda)},$$

where δ_v is the metric on T_v , d is the degree functor for the k -graph Λ , and $|n| = \sum_{i=1}^k n_i$ for $n = (n_1, \dots, n_k) \in \mathbb{N}^k$. Then there exists a unique k -surjective Λ -subsystem (K, ψ) such that each K_v is a bounded closed subset of T_v , and in fact each K_v is compact.

Note that to check the hypothesis it suffices to show that each of the generating maps σ_λ for $\lambda \in \Lambda^{e_i}$ has Lipschitz constant at most c , where e_1, \dots, e_k is the standard basis for \mathbb{N}^k .

Proof. Let

$$\mathcal{C} = \prod_{v \in \Lambda^0} \mathcal{C}(T_v),$$

where for $v \in \Lambda^0$ we let $\mathcal{C}(T_v)$ denote the set of bounded closed subsets of T_v , which is complete under the Hausdorff metric. Since Λ^0 is countable, \mathcal{C} is a complete metric space. For each $n \in \mathbb{N}^k$ define a map $\tilde{\sigma}^n : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\tilde{\sigma}^n(C)_v = \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(C_{s(\lambda)}).$$

As in [MW88], $\tilde{\sigma}^n$ is a contraction, and so has a unique fixed point in \mathcal{C} . We need to know that the maps $\{\tilde{\sigma}^n\}_{n \in \mathbb{N}^k}$ all have the same fixed point, and it suffices to show that they commute. Let $n, m \in \mathbb{N}^k$. Then for all $C = (C_v)_{v \in \Lambda^0} \in \mathcal{C}$ and $v \in \Lambda^0$ we have

$$\begin{aligned} \tilde{\sigma}^n \circ \tilde{\sigma}^m(C)_v &= \tilde{\sigma}^n(\tilde{\sigma}^m(C))_v \\ &= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda(\tilde{\sigma}^m(C)_{s(\lambda)}) \\ &= \bigcup_{\lambda \in v\Lambda^n} \sigma_\lambda \left(\bigcup_{\mu \in s(\lambda)\Lambda^m} \sigma_\mu(C_{s(\mu)}) \right) \\ &= \bigcup_{\lambda \in v\Lambda^n} \bigcup_{\mu \in s(\lambda)\Lambda^m} \sigma_\lambda \circ \sigma_\mu(C_{s(\mu)}) \\ &= \bigcup_{\lambda\mu \in v\Lambda^{n+m}} \sigma_{\lambda\mu}(C_{s(\lambda\mu)}) \\ &= \bigcup_{\alpha \in v\Lambda^{n+m}} \sigma_\alpha(C_{s(\alpha)}), \end{aligned}$$

which, by the factorization property of Λ , coincides with

$$\bigcup_{\mu \in v\Lambda^m} \bigcup_{\lambda \in s(\mu)\Lambda^n} \sigma_\mu \circ \sigma_\lambda(C_{s(\lambda)}) = \tilde{\sigma}^m \tilde{\sigma}^n(C)_v.$$

Letting $(K_v)_{v \in \Lambda^0}$ be the unique common fixed point of $\tilde{\sigma}$ on \mathcal{C} , we see that, setting $K = \bigcup_{v \in \Lambda^0} K_v$ and $\psi = \sigma|_K$, the restriction (K, ψ) of (T, σ) is the unique k -surjective Λ -subsystem with bounded closed subsets K_v .

To see that in fact every K_v is compact, play the same game with $\mathcal{C}(T_v)$ replaced by the set of compact subsets of T_v , again getting a

unique fixed point. But since the compact subsets are among the bounded closed subsets, the resulting Λ -subsystem must coincide with the one we found above, by uniqueness. \square

Definition 4.2. A *Mauldin-Williams Λ -system* is a k -surjective Λ -system of maps (T, σ) such that each T_v is a compact metric space and, for some $c < 1$, every $\sigma_\lambda : T_{s(\lambda)} \rightarrow T_{r(\lambda)}$ is a contraction with Lipschitz constant at most $c^{|d(\lambda)|}$.

Proposition 4.3. *Every Mauldin-Williams Λ -system (T, σ) is self-similar, and if X is the associated Λ -system of correspondences then $\mathcal{O}_X \cong C^*(\Lambda)$.*

Proof. We adapt the technique of Ionescu [Ion07]. Let $x \in v\Lambda^\infty$, so that $x : \Omega_k \rightarrow \Lambda$ is a k -graph morphism. For each $n \in \mathbb{N}^k$ let $x(0, n)$ be the unique path $\lambda \in \Lambda^n$ such that $x = \lambda y$ for some (unique) $y \in s(\lambda)\Lambda^\infty$. By definition of Mauldin-Williams Λ -system, the range of each $\sigma_{x(0, n)}$ has diameter at most $c^{|n|}$. Thus by compactness there is a unique element $\Phi(x) \in T_v$ such that

$$\bigcap_{n \in \mathbb{N}^k} \sigma_{x(0, n)}(T_{s(x(0, n))}) = \{\Phi(x)\}.$$

We get a map $\Phi : \Lambda^\infty \rightarrow T$, which is continuous because for each $x \in \Lambda^\infty$ the images under Φ of the neighborhoods $x(0, n)\Lambda^\infty$ of x have diameters shrinking to 0. By construction, it's obvious that

$$\Phi(\lambda x) = \sigma_\lambda(\Phi(x)) \quad \text{for all } \lambda \in \Lambda, x \in s(\lambda)\Lambda^\infty,$$

so Φ is intertwining.

We show that Φ is surjective. Put $T' = \Phi(\Lambda^\infty)$. By Proposition 3.22, $(T', \sigma|_{T'})$ is k -surjective with each T'_v compact, which implies that $T' = T$ by the uniqueness in Proposition 4.1.

Finally, it now follows from Theorem 3.23 that $\mathcal{O}_X \cong C^*(\Lambda)$. \square

Remark 4.4. It would be completely routine at this point to adapt Ionescu's techniques to prove a higher-rank version his other “no-go theorem” [Ion07, Theorem 3.4], namely that there are no “noncommutative Mauldin-Williams Λ -systems” of maps.

5. THE ASSOCIATED TOPOLOGICAL k -GRAPH

Let Λ be a row-finite k -graph with no sources, and let (T, σ) be a k -regular Λ -system of maps. We do *not* assume that (T, σ) is self-similar unless otherwise noted.

Let (T, σ) be a Λ -system of maps. We want to define a topological k -graph $\Lambda * T$ as follows:

- (1) $\Lambda * T = \{(\lambda, t) \in \Lambda \times T : t \in T_{s(\lambda)}\}$;
- (2) $s(\lambda, t) = (s(\lambda), t)$ and $r(\lambda, t) = (r(\lambda), \sigma_\lambda(t))$;
- (3) if $s(\lambda) = r(\mu)$ and $t = \sigma_\mu(s)$, then $(\lambda, t)(\mu, s) = (\lambda\mu, s)$;
- (4) $d(\lambda, t) = d(\lambda)$.

$\Lambda * T$ has the relative topology from $\Lambda \times T$, and is the disjoint union of the open subsets $\{\lambda\} \times T_{s(\lambda)}$, each of which is a homeomorphic copy of $T_{s(\lambda)}$.

Proposition 5.1. *The above operations make $\Lambda * T$ into a topological k -graph.*

Proof. This is routine. For instance, it's completely routine to check that $\Lambda * T$ is a small category and the map defined in (4) is a functor. Let's check the unique factorization property: Let $(\lambda, t) \in \Lambda * T$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$. Then we can uniquely write $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$. We have

$$(\lambda, t) = (\mu, \sigma_\nu(t))(\nu, t), \quad d(\mu, \sigma_\nu(t)) = m, \quad \text{and} \quad d(\nu, t) = n,$$

and $(\mu, \sigma_\nu(t))$ and (ν, t) are unique since μ and ν are. It's immediate that the degrees match up and this factorization is unique.

The multiplication on the category $\Lambda * T$ is continuous and open because it is in fact a local homeomorphism from the fibred product $(\Lambda * T) * (\Lambda * T)$ to $\Lambda * T$, which for each $(\lambda, \mu) \in \Lambda \times \Lambda$ with $s(\lambda) = r(\mu)$ maps the open subset

$$\left((\{\lambda\} \times T_{s(\lambda)}) \times (\{\mu\} \times T_{s(\mu)}) \right) \cap \left((\Lambda * T) * (\Lambda * T) \right)$$

bijectionally onto the open subset $\{\lambda\mu\} \times T_{s(\mu)}$.

To see that the source map on $\Lambda * T$ is a local homeomorphism, just note that it restricts to homeomorphisms

$$\{\lambda\} \times T_{s(\lambda)} \rightarrow \{s(\lambda)\} \times T_{s(\lambda)}. \quad \square$$

Remark 5.2. One could reasonably regard a Λ -system of maps as an action of Λ on the space $T = \bigsqcup_{v \in \Lambda^0} T_v$, and the topological k -graph $\Lambda * T$ as the associated transformation k -graph.

Remark 5.3. If each T_v is discrete and every map $\sigma_\lambda : T_{s(\lambda)} \rightarrow T_{r(\lambda)}$ is bijective, then the above k -graph $\Lambda * T$ coincides with that of [PQR05, Proposition 3.3], where the main point was that the coordinate projection $(\lambda, t) \mapsto \lambda$ is a model for coverings of the k -graph Λ .

Proposition 5.4. *Let (T, σ) be a k -regular Λ -system of maps, and let (\mathcal{A}, φ) be the associated Λ -system of homomorphisms, which in turn has an associated Λ -system X of correspondences. Then*

$$\mathcal{O}_X \cong C^*(\Lambda * T),$$

where $\Lambda * T$ is the topological k -graph of Proposition 5.1.

Proof. Our strategy is to show that \mathcal{O}_X and $C^*(\Lambda * T)$ are isomorphic to the Cuntz-Pimsner algebras of isomorphic \mathbb{N}^k -systems of correspondences. Recall that $\mathcal{O}_X \cong \mathcal{O}_Y$, where $Y = Y_X$ is the \mathbb{N}^k system associated to X . Thus for each $n \in \mathbb{N}^k$ we have

$$Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda,$$

where X_λ is the correspondence over $A = \bigoplus_{v \in \Lambda^0} A_v$ naturally associated (via identifying the A_v 's with direct summands in A) to the standard $A_{r(\lambda)} - A_{s(\lambda)}$ correspondence $\sigma_\lambda^* A_{s(\lambda)}$ determined by the homomorphism $\sigma_\lambda^* : A_{r(\lambda)} \rightarrow M(A_{s(\lambda)})$ given by composition with $\sigma_v : T_{s(v)} \rightarrow T_{r(v)}$.

On the other hand, by [CLSV11, Theorem 5.20] $C^*(\Lambda * T)$ is isomorphic to the Cuntz-Nica-Pimsner algebra \mathcal{NO}_Z , where Z is the \mathbb{N}^k -system of $C_0((\Lambda * T)^0)$ -correspondences associated to the topological k -graph $\Lambda * T$. As we'll show in this proof, the \mathbb{N}^k -systems Z and Y are isomorphic. Since the Λ -system (T, σ) is k -regular, so is Y , and hence so is Z . In particular, since each pair in \mathbb{N}^k has an upper bound, and $C_0((\Lambda * T)^0)$ maps injectively into the compacts on Z_n for every $n \in \mathbb{N}^k$, it follows from [SY10, Corollary 5.2] that $\mathcal{NO}_Z = \mathcal{O}_Z$, because by [Fow02, Proposition 5.8] Z is compactly aligned.

Let's see what the Λ -system Z looks like in this situation: for each $n \in \mathbb{N}^k$, the correspondence Z_n over $C_0((\Lambda * T)^0)$ is a completion of $C_c((\Lambda * T)^n)$. We can safely identify $(\Lambda * T)^0$ with $T = \bigsqcup_{v \in \Lambda^0} T_v$, and hence $C_0((\Lambda * T)^0)$ with $A = \bigoplus_{v \in \Lambda^0} C_0(T_v)$, and in this way Z_n becomes an A -correspondence. For $\xi, \eta \in C_c((\Lambda * T)^n) = C_c(\Lambda^n * T)$, the inner product is given by

$$\langle \xi, \eta \rangle_A(t) = \sum_{\lambda \in \Lambda^{n_v}} \overline{\xi(\lambda, t)} \eta(\lambda, t), \quad t \in T_v, v \in \Lambda^0,$$

and the right and left module operations are given for $f \in A$ by

$$\begin{aligned} (\xi \cdot f)(\lambda, t) &= \xi(\lambda, t) f(t) \\ (f \cdot \xi)(\lambda, t) &= f(\sigma_\lambda(t)) \xi(\lambda, t). \end{aligned}$$

Note that

$$(\Lambda * T)^n = \bigsqcup_{\lambda \in \Lambda^n} (\{\lambda\} \times T_{s(\lambda)}).$$

Thus for each $\lambda \in \Lambda^n v$ we have a natural inclusion map

$$C_c(\{\lambda\} \times T_v) \hookrightarrow Z_n,$$

and Z_n is the closed span of these subspaces. Moreover, their closures form a pairwise orthogonal family of subcorrespondences of Z_n :

$$Z_n(\lambda) = \overline{C_c(\{\lambda\} \times T_v)} \quad \text{for } \lambda \in \Lambda^n v,$$

and we see that

$$Z_n = \bigoplus_{\lambda \in \Lambda^n} Z_n(\lambda)$$

as A -correspondences.

We will obtain an isomorphism $\psi : Y \rightarrow Z$ of \mathbb{N}^k -systems by defining isomorphisms $\psi_n : Y_n \rightarrow Z_n$ of A -correspondences and then verifying that

$$\psi_n(\xi)\psi_m(\eta) = \psi_{n+m}(\xi\eta) \quad \text{for all } (\xi, \eta) \in Y_n \times Y_m.$$

By the above, to get an isomorphism $\psi_n : Y_n \rightarrow Z_n$ it suffices to get isomorphisms $\psi_{n,\lambda} : X_\lambda \rightarrow Z_n(\lambda)$ for each $\lambda \in \Lambda^n$. If $\lambda \in \Lambda^n v$ and

$$\xi \in C_c(T_v) \subset X_\lambda$$

define

$$\psi(\xi) \in C_c(\{\lambda\} \times T_v) \subset Z_n(\lambda)$$

by

$$\psi(\xi)(\lambda, t) = \xi(t).$$

Routine computations show that $\psi_{n,\lambda}$ is an isomorphism.

Now we check multiplicativity, and again it suffices to consider the fibres of the Λ -system X : if

$$\begin{aligned} \xi &\in X_\lambda \quad \text{for } \lambda \in \Lambda^n v \\ \eta &\in X_\mu \quad \text{for } \mu \in v\Lambda^m \end{aligned}$$

then for $t \in T_{s(\mu)}$ we have

$$\begin{aligned} (\psi_{n,\lambda}(\xi)\psi_{m,\mu}(\eta))(\lambda\mu, t) &= \psi_{n,\lambda}(\xi)(\lambda, \sigma_\mu(t))\psi_{m,\mu}(\eta)(\mu, t) \\ &= \xi(\sigma_\mu(t))\eta(t) \\ &= (\xi\eta)(t) \\ &= (\psi_{n+m,\lambda\mu}(\xi\eta))(\lambda\mu, t). \end{aligned} \quad \square$$

6. THE TENSOR GROUPOIDS

Recall that in [FS02] Fowler and Sims study what they call *product systems taking values in a tensor groupoid*. Their product systems are over semigroups, and here we want to consider the special cases related to our Λ -systems of homomorphisms or maps, where the k -graph Λ has a single vertex, and so in particular is a monoid whose identity element is the unique vertex. Since we won't need to do serious work with the concept, here we informally regard a *tensor groupoid* as a groupoid \mathcal{G}

with a “tensor” operation $X \otimes Y$ and an “identity” object $1_{\mathcal{G}}$ such that the “expected” redistributions of parentheses and canceling of tensoring with the identity are implemented via given natural equivalences. As defined in [FS02], a *product system* over a semigroup S taking values in a tensor groupoid \mathcal{G} is a family $\{X_s\}_{s \in S}$ of objects in \mathcal{G} together with an associative family $\{\alpha_{s,t}\}_{s,t \in S}$ of isomorphisms

$$\alpha_{s,t} : X_s \otimes X_t \rightarrow X_{st},$$

and moreover if S has an identity e then $X_e = 1_{\mathcal{G}}$ and $\alpha_{e,s}, \alpha_{s,e}$ are the given isomorphisms $1_{\mathcal{G}} \otimes X_s \cong X_s$ and $X_s \otimes 1_{\mathcal{G}} \cong X_s$.

Systems of homomorphisms. Let A be a C^* -algebra, and \mathcal{G} be the tensor groupoid whose objects are the nondegenerate homomorphisms $\pi : A \rightarrow M(A)$, whose only morphisms are the identity morphisms on objects, and with identity $1_{\mathcal{G}} = \text{id}_A$. Define a tensor operation on \mathcal{G} by composition:

$$\pi_1 \otimes \pi_2 = \pi_2 \circ \pi_1,$$

where π_2 has been canonically extended to a strictly continuous unital endomorphism of $M(A)$. Standard properties of composition show that \mathcal{G} is indeed a tensor groupoid, in a trivial way: the tensor operation is actually associative, and $1_{\mathcal{G}}$ acts as an actual identity for tensoring, so the axioms of [FS02] for a tensor groupoid are obviously satisfied.

Due to the special nature of this tensor groupoid \mathcal{G} , a *product system over \mathbb{N}^k taking values in \mathcal{G}* , as in [FS02, Definition 1.1], is a homomorphism $n \mapsto \varphi_n$ from the additive monoid \mathbb{N}^k into the monoid of nondegenerate homomorphisms $A \rightarrow M(A)$ under composition, in other words such a product system is precisely what we call in the current paper an \mathbb{N}^k -system of homomorphisms.

Systems of maps. Quite similarly to the above, let T be a locally compact Hausdorff space, and \mathcal{G} be the tensor groupoid whose objects are the continuous maps $\sigma : X \rightarrow X$, whose only morphisms are the identity morphisms on objects, and with identity $1_{\mathcal{G}} = \text{id}_X$. Define a tensor operation on \mathcal{G} by composition:

$$\sigma \otimes \psi = \sigma \circ \psi.$$

Again, \mathcal{G} is indeed a tensor groupoid, in a trivial way, because the tensor operation is actually associative, and $1_{\mathcal{G}}$ acts as an actual identity for tensoring.

A *product system over \mathbb{N}^k taking values in \mathcal{G}* , as in [FS02, Definition 1.1], is a homomorphism $n \mapsto \sigma_n$ from the additive monoid \mathbb{N}^k into the monoid of continuous selfmaps of X maps under composition,

in other words such a product system is precisely what we call in the current paper an \mathbb{N}^k -system of maps.

7. REVERSING THE PROCESSES

In Remark 3.2 we noted that every Λ -system of maps gives rise to a Λ -system of homomorphisms, and every Λ -system of homomorphisms gives rise to a Λ -system of correspondences. In this section we will investigate the extent to which these two processes are reversible.

Question 7.1. When is a given Λ -system of correspondences isomorphic to the one associated to a Λ -system of homomorphisms?

Investigating this question requires us to examine balanced tensor products of standard correspondences. First we observe without proof the following elementary fact.

Lemma 7.2. *Let $\varphi : A \rightarrow M(B)$ and $\psi : B \rightarrow M(C)$ be nondegenerate homomorphisms. Then there is a unique $A - C$ correspondence isomorphism*

$$\theta : {}_{\varphi}B \otimes_B {}_{\psi}C \xrightarrow{\cong} {}_{\psi \circ \varphi}C$$

such that

$$\theta(b \otimes c) = \psi(b)c \quad \text{for } b \in B, c \in C.$$

We can analyze the question of whether a given Λ -system X of correspondences is isomorphic to one coming from a Λ -system of homomorphisms in several steps:

First of all, without loss of generality we can look for a Λ -system of homomorphisms of the form (\mathcal{A}, φ) .

Next, for each $\lambda \in u\Lambda v$ the $A_u - A_v$ correspondence X_λ must be isomorphic to a standard one, more precisely there must exist a linear bijection

$$\theta_\lambda : X_\lambda \rightarrow A_v$$

and a nondegenerate homomorphism

$$\varphi_\lambda : A_u \rightarrow M(A_v)$$

such that

$$(7.1) \quad \theta_\lambda(\xi)^* \theta_\lambda(\eta) = \langle \xi, \eta \rangle_{A_v} \quad \text{for all } \xi, \eta \in X_\lambda$$

$$(7.2) \quad \theta_\lambda(a \cdot \xi \cdot b) = \varphi_\lambda(a) \theta_\lambda(\xi) b \quad \text{for all } a \in A_u, \xi \in X_\lambda, b \in A_v.$$

Moreover, whenever $\lambda \in u\Lambda v, \mu \in v\Lambda w$ we must have

$$\begin{aligned} {}_{\varphi_{\lambda\mu}}A_w &= X_{\lambda\mu} \\ &\cong X_\lambda \otimes_{A_v} X_\mu \end{aligned}$$

$$\begin{aligned}
 &= \varphi_\lambda A_v \otimes_{A_v} \varphi_\mu A_w \\
 &\cong_{\varphi_\mu \circ \varphi_\lambda} A_w,
 \end{aligned}$$

so there exists a unitary multiplier $U(\lambda, \mu) \in M(A_w)$ such that

$$\varphi_\mu \circ \varphi_\lambda = \text{Ad } U(\lambda, \mu) \circ \varphi_{\lambda\mu}.$$

The $U(\lambda, \mu)$'s satisfy a kind of “two-cocycle” identity coming from associativity of composition of the φ_λ 's.

Now, if this Λ -system of correspondences is isomorphic to one associated to a Λ -system (\mathcal{A}, ψ) of homomorphisms, then for each $\lambda \in u\Lambda v$ we must have an isomorphism $\varphi_\lambda A_v \cong \psi_\lambda A_v$ of $A_u - A_v$ correspondences, and so there must be a unitary multiplier $W_\lambda \in M(A_v)$ such that

$$\varphi_\lambda = \text{Ad } W_\lambda \circ \psi_\lambda.$$

Since (\mathcal{A}, ψ) is a Λ -system of homomorphisms, whenever $\lambda \in u\Lambda v, \mu \in v\Lambda w$ we have

$$\begin{aligned}
 \varphi_{\lambda\mu} &= \text{Ad } W_{\lambda\mu} \circ \psi_{\lambda\mu} \\
 &= \text{Ad } W_{\lambda\mu} \circ \psi_\mu \circ \psi_\lambda \\
 &= \text{Ad } W_{\lambda\mu} \circ \text{Ad } W_\mu^* \circ \varphi_\mu \circ \text{Ad } W_\lambda^* \circ \varphi_\lambda \\
 &= \text{Ad } W_{\lambda\mu} W_\mu^* \varphi_\mu(W_\lambda^*) \circ \varphi_\mu \circ \varphi_\lambda \\
 &= \text{Ad } W_{\lambda\mu} W_\mu^* \varphi_\mu(W_\lambda^*) U(\lambda, \mu) \circ \varphi_{\lambda\mu},
 \end{aligned}$$

so since the homomorphisms φ_λ are nondegenerate we see that, in the quotient group of the unitary multipliers of A_w modulo the central unitary multipliers, the cosets satisfy

$$[U(\lambda, \mu)] = [\varphi_\mu(W_\lambda) W_\mu W_{\lambda\mu}^*],$$

giving a sort of cohomological obstruction (which we won't make precise) to the Λ -system of correspondences being isomorphic to a one associated to a Λ -system (\mathcal{A}, ψ) of homomorphisms.

Note that if all the C^* -algebras A_v are commutative, then none of the above unitary multipliers appear, so once we have θ_λ 's and φ_λ 's satisfying (7.1) then the pair (\mathcal{A}, φ) will automatically be a Λ -system of homomorphisms whose associated Λ -system of correspondences is isomorphic to X . What makes this happen is the way in which the correspondences X_λ fit together. This is worth recording:

Proposition 7.3. *Let X be a Λ -system of correspondences such that every A_v is commutative. Then X is isomorphic to the Λ -system associated to a Λ -system of homomorphisms if and only if, whenever $\lambda \in u\Lambda v$, X_λ is isomorphic to a standard $A_u - A_v$ correspondence $\varphi_\lambda A_v$.*

Proposition 7.4. *Let (\mathcal{A}, φ) be a Λ -system of homomorphisms such that every A_v is commutative, and for each $v \in \Lambda^0$ let T_v be the maximal ideal space of A_v . Then there is a unique Λ -system of maps (T, σ) such that (\mathcal{A}, φ) is the associated Λ -system of homomorphisms.*

On the other hand, every Λ -system of homomorphisms is uniquely isomorphic to the one associated to a Λ -system of maps, at least in the only circumstances where it makes sense:

Proof. This follows immediately from the duality between the category of commutative C^* -algebras and nondegenerate homomorphisms into multiplier algebras and the category of locally compact Hausdorff spaces and continuous maps. \square

8. NO HIGHER-RANK FRACTALS

In Proposition 3.22 we showed that every Λ system of maps (T, σ) has a self-similar k -surjective Λ -subsystem $(T', \sigma|_{T'})$. The self-similar set T' is the part of the system that would generally be referred to as the “fractal”. It is natural to wonder whether the generalization to k -graphs presented here gives rise to any new fractals that could not have arisen from the corresponding constructions for 1-graphs. The answer to this question turns out to be “no” for reasons we will now explain. Throughout the following discussion, let $p = (1, 1, \dots, 1) \in \mathbb{N}^k$

Definition 8.1. For a k -graph Λ we define the *diagonal 1-graph* E as follows:

$$\begin{aligned} E^0 &= \Lambda^0 \\ E^1 &= \{e_\lambda : \lambda \in \Lambda, d(\lambda) = p\} \\ r(e_\lambda) &= r(\lambda) \\ s(e_\lambda) &= s(\lambda). \end{aligned}$$

If (T, σ) is a Λ -system of maps, then we define the *diagonal E -system* (T, ρ) of (T, σ) to be the E -system of maps such that $\rho_{e_\lambda} = \sigma_\lambda$ for all $e_\lambda \in E^1$. Finally, let $\alpha : E^* \rightarrow \Lambda$ be the map defined by $\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Proposition 8.2. *The map $i : \Lambda^\infty \rightarrow E^\infty$ defined by $\alpha(i(x)(j, l)) = x(jp, lp)$ is a bijection and i^{-1} is continuous.*

Proof. First we must show that this is well-defined. This just amounts to showing that α is injective. To see this recall that if $\alpha(e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}) = \lambda$ then $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$ where each λ_i has degree p and hence $d(\lambda_1 \lambda_2 \cdots \lambda_n) = np$. Since there is only one way to write np

as a sum of p 's, there is only one such decomposition of λ (by unique factorization), so if $\alpha(e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_n}) = \alpha(e_{\gamma_1}e_{\gamma_2}\cdots e_{\gamma_n})$ we must have $\lambda_i = \gamma_i$ for all i .

Next, to show that i is injective, suppose $i(x) = i(y)$ for $x, y \in \Lambda^\infty$. Then by definition we must have that $x(jp, lp) = y(jp, lp)$ for all $j, l \in \mathbb{N}$, and in particular we have that $x(0, jp) = y(0, jp)$ for all $j \in \mathbb{N}$. But since $\{jp\}_j$ is a cofinal increasing sequence in \mathbb{N}^k , x and y are uniquely determined by their values on the pairs $(0, jp)$ (see [KP, Remarks 2.2]) so we must have $x = y$.

Now, to show that i is surjective, let $z \in E^\infty$. We wish to find an infinite path $x \in \Lambda^\infty$ such that $i(x) = z$. We will again make use of the fact that such an x is uniquely determined by its values on $(0, jp)$. Specifically, if $z(0, j) = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_j}$, then we let $x(0, jp) = \lambda_1\lambda_2\cdots\lambda_j$. Then $\alpha(i(x)(0, j)) = x(0, jp) = \lambda_1\lambda_2\cdots\lambda_j$ and $\alpha(z(0, j)) = \lambda_1\lambda_2\cdots\lambda_j$ so by the injectivity of α we have that $i(x)(0, j) = z(0, j)$ and since $i(x)$ and z are uniquely determined by their values at $(0, j)$ we have that $i(x) = z$.

Finally, we need to show that i^{-1} is continuous. We have

$$\alpha(i(x)(j, l)) = x(jp, lp) = \lambda_j\cdots\lambda_l,$$

where $\lambda_j\cdots\lambda_l$ is the unique decomposition of $x(jp, lp)$ into paths of degree p . Since α is injective, we get $i(x)(j, l) = e_{\lambda_j}\cdots e_{\lambda_l}$. Since this holds for all (j, l) we must have that $i^{-1}(e_{\lambda_1}e_{\lambda_2}\cdots) = \lambda_1\lambda_2\cdots$ for all $e_{\lambda_1}e_{\lambda_2}\cdots \in E^\infty$. Recall that the topologies on E^∞ and Λ^∞ are generated by the collections $\{Z(P) : P \in E^*\}$ and $\{Z(\lambda) : \lambda \in \Lambda\}$ respectively where $Z(P) = \{Pz : z \in s(P)E^\infty\}$ and $Z(\lambda) = \{\lambda x : x \in s(\lambda)\Lambda^\infty\}$. Thus a net $\{\lambda_1^\alpha\lambda_2^\alpha\cdots\}_{\alpha \in A}$ in Λ^∞ converges to $\lambda_1\lambda_2\cdots$ in Λ^∞ if for all $n \in \mathbb{N}$ there is $\alpha_0 \in A$ such that $\lambda_j^\alpha = \lambda_j$ for all $j \leq n$ and $\alpha \geq \alpha_0$, and similarly for nets in E^∞ . Now, suppose $\{e_{\lambda_1^\alpha}e_{\lambda_2^\alpha}\cdots\}_{\alpha \in A}$ converges to $e_{\lambda_1}e_{\lambda_2}\cdots$ in E^∞ . Then for all $n \in \mathbb{N}$ there is $\alpha_0 \in A$ such that $e_{\lambda_j^\alpha} = e_{\lambda_j}$ for all $j \leq n$ and $\alpha \geq \alpha_0$. Thus $\lambda_j^\alpha = \lambda_j$ for all $j \leq n$ and $\alpha \geq \alpha_0$, and we have shown that the net $\{i^{-1}(e_{\lambda_1^\alpha}e_{\lambda_2^\alpha}\cdots)\}_{\alpha \in A} = \{\lambda_1^\alpha\lambda_2^\alpha\cdots\}_{\alpha \in A}$ converges to $i^{-1}(e_{\lambda_1}e_{\lambda_2}\cdots) = \lambda_1\lambda_2\cdots$ in Λ^∞ . Therefore i^{-1} is continuous. \square

Proposition 8.3. *Let (T, σ) be a Λ -system of maps and let (T, ρ) be the diagonal E -system of (T, σ) . If $\Phi : \Lambda^\infty \rightarrow T$ is intertwining with respect to (T, σ) then $\Phi \circ i^{-1} : E^\infty \rightarrow T$ is intertwining with respect to (T, ρ) .*

Proof. We have:

$$\Phi \circ i^{-1} \circ \tau_{e_\lambda}(x) = \Phi(i^{-1}(e_\lambda x)) = \Phi(\lambda i^{-1}(x)) = \Phi \circ \tau_\lambda(i^{-1}(x))$$

but since Φ is intertwining, this gives:

$$= \sigma_\lambda \circ \Phi(i^{-1}(x)) = \rho_{e_\lambda} \circ \Phi \circ i^{-1}(x).$$

Since x was arbitrary, we have $(\Phi \circ i^{-1}) \circ \tau_\lambda = \rho_\lambda \circ (\Phi \circ i^{-1})$ so $\Phi \circ i^{-1}$ is intertwining with respect to (T, ρ) . \square

Definition 8.4. If (T, σ) is a Λ system of maps, Φ is an intertwining map, and $(T', \sigma|_{T'})$ is the self-similar k -surjective Λ -subsystem of Proposition 3.22, then we call T' the *attractor* of (T, σ, Φ) .

Theorem 8.5. *Let Λ be a k -graph. Suppose (T, σ) is a Λ -system of maps, Φ is an intertwining map with respect to (T, σ) , and T' is the attractor of (T, σ, Φ) . Then there exist a 1-graph E with $E^0 = \Lambda^0$, an E -system of maps (T, ρ) , and an intertwining map Ψ with respect to (T, ρ) such that if T'' is the attractor of (T, ρ, Ψ) then $T'' = T'$.*

Proof. Let E be the diagonal 1-graph of Λ , (T, ρ) be the diagonal E -system of (T, σ) , and $\Psi = \Phi \circ i^{-1}$. Proposition 8.3 shows that this is an intertwining map. For all $v \in \Lambda^0$ we have

$$T''_v = \Psi(vE^\infty) = \Phi(i^{-1}(vE^\infty)) = \Phi(v\Lambda^\infty) = T'_v,$$

and hence $T'' = T'$. \square

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